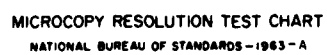END

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ROBUST RANGE MEASUREMENT PREPROCESSING

TECHNICAL REPORT NO. 83

OCTOBER 1982

MATHEMATICAL SERVICES BRANCH
DATA SCIENCES DIVISION
US ARMY WHITE SANDS MISSILE RANGE
WHITE SANDS MISSILE RANGE, NEW MEXICO

ROBUST RANGE MEASUREMENT PREPROCESSING

TECHNICAL REPORT NO. 83

OCTOBER 1982

Written by: _____
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Mathematician

_____
ROBERT H. TURNER
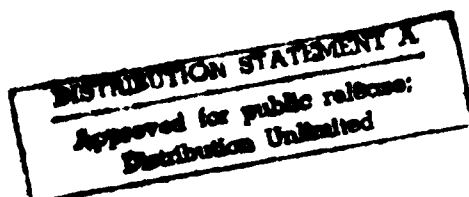Mathematician

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER TECHNICAL REPORT NO. 83	2. GOVT ACCESSION NO. AD-A121463	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ROBUST RANGE MEASUREMENT PREPROCESSING		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) WILLIAM S. AGEE ROBERT H. TURNER		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Data Sciences Division National Range Operations Directorate White Sands Missile Range, New Mexico 88002		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematical Services Branch (STEWS-NR-AM) Data Sciences Division, National Range Operations White Sands Missile Range, NM 88002		12. REPORT DATE OCTOBER 1982
		13. NUMBER OF PAGES 27
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A

16. DISTRIBUTION STATEMENT (of this Report)

DISTRIBUTION OF THIS DOCUMENT IN UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Robust Estimation, Iteratively reweighted least squares, cubic spline fit, Range measuring system, outliers

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
The RMS/MTTS instrumentation system located at MacGregor Range is a range measuring, multiple target tracking system. In order to obtain a vehicle trajectory from this system, the range measurement from several receivers are processed by least squares. Since the measured vehicle trajectories are often low altitude, the resulting nonlinear least squares equations are ill-conditioned. In addition, this measurement system is plagued by outliers, sometimes by dense burst of outliers. The combination of ill-conditioning and outliers is lethal
(Continued on Reverse)

20. ABSTRACT (CONTINUED)

and attempts to robustify the nonlinear least squares processing have failed. An alternative is to preprocess each of the range measurement sequences, eliminating the outliers and replacing them if necessary. Each sequence of range measurements is preprocessed by robustly fitting a cubic spline using iteratively reweighted least squares. Due to the nature of spline fitting and the possible dense bursts of outliers, the choice of a good set of initial weights for use in the iteratively reweighted least squares is important to the efficiency of the method. These initial weights are determined using robust, local fitting techniques. Several robust techniques have been tested for this local fitting application. The robust spline preprocessing is illustrated with some especially troublesome data sequences and the relative performance of several robust methods for choosing the initial weights is compared.

TABLE OF CONTENTS

INTRODUCTION

The RMS/MTTS instrumentation system located at MacGregor Range is a range measuring, multiple target tracking system. In order to obtain a vehicle trajectory from this system, the range measurements from several receivers are processed by least squares. Because the measured vehicle trajectories of interest are often low altitude and because of the geometry of the receiving stations, the resulting nonlinear least squares equations are often ill-conditioned. In addition, this measurement system is subject to outliers, sometimes dense bursts of outliers. This combination of ill-conditioning and outliers is lethal and our attempts to robustify the nonlinear least squares estimation process have failed. An alternative is to preprocess each of the range measurement sequences, identifying the outliers, and replacing them if necessary. The ill-conditioned least squares problem can then be treated without being troubled by outliers.

Suppose we preprocess the measurement sequence, $R(t_i)$, $i = 1,N$. For typical aircraft trajectories the measurement rate is 10/sec with a time of interest of 40 - 120 sec so that N is often in the range 400 - 1200. The purpose of the preprocessing may be to detect outliers, to precompute measurement variances for future least squares processing, or to synchronize several different discrete measurement sequences. The preprocessing of the range measurement sequence, $R(t_i)$, is done by fitting a cubic spline to the discrete measurements using iteratively reweighted least squares (IRWLS). Specifically, at the kth iteration we minimize,

$$\sum_{j=1}^{N} w_j^{(k)} \left(R(t_j) - \sum_i b_i^{(k)} B_i(t_j)\right)^2, \tag{1}$$

where $B_i(\cdot)$ are the cubic B-splines and $b_i^{(k)}$ are the spline coefficients to be estimated. The weights, $W_j^{(k)}$ are computed from the Hampel ψ-function using the spline fit from the $(k-1)\underline{\text{st}}$ iteration.

$$W_j^{(k)} = \frac{\psi\left(\frac{R(t_j) - \sum_i b_i^{(k-1)} B_i(t_j)}{S_j^{k-1}}\right)}{\frac{R(t_j) - \sum_i b_i^{(k-1)} B_i(t_j)}{S_j^{k-1}}} \tag{2}$$

where

$$\psi(x) = \begin{cases} x & |x| \leq a \\ a \cdot \text{sgn}\ (x) & a < |x| \leq b \\ a\left(\frac{x-c\ \text{sgn}(x)}{b-c}\right) & b < |x| \leq c \\ 0 & |x| > c \end{cases} \tag{3}$$
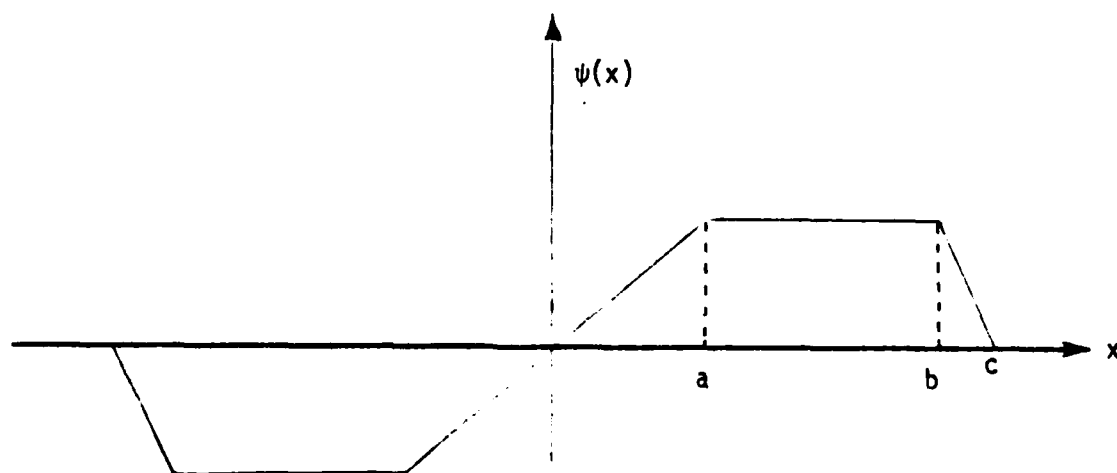
$S_j^{(k-1)}$ estimates the dispersion in the residual, $R(t_j) - \sum_i b_i^{(k-1)} B_i(t_j)$. The value of $S_j^{(k-1)}$ can be computed either locally or globally from the residuals at the $(k-1)\underline{st}$ iteration. The dispersion $S_j^{(k-1)}$ is a MAD estimate obtained from

$$S_j^{(k-1)} = \operatorname*{median}_{m \in T_j} |R(t_m) - \sum_i b_i^{(k-1)} B_i(t_m)|/.6745 \tag{4}$$

If the set T_j is in some sense the set of points close to t_j, the estimate $S_j^{(k-1)}$ is local. If the set T_j is the set, $T_j = \{t_m | m = 1,N\}$ the estimate is global. For the present application only the global estimate $S_j^{(k-1)} = S^{(k-1)}$ will be used. If a very long data sequence, say about one hour, a local estimate would probably be preferable to the global estimate.

CHOICE OF KNOTS

Let $\{T_i, i = 1,M\}$ be a set of knot times. These knot times are used to define the cubic B-splines, $B_i(t_j)$. Of most importance in the choice of the knot times is their spacing, which determines the ability of the cubic spline to fit the data. However, for each additional knot time there is one additional spline co-efficient to be estimated, thus increasing the computational load. Thus, we want to have as few knots as possible and the rules for their choice simple and yet be able to adequately represent the data. With this simple philosophy for selecting knots we will try to assign a fixed number of data points, NPTO, to each knot interval. The first four knots are placed at the first data. The last knot interval may have more than NPTO points but fewer than 2 · NPTO data points. If there is a large time break in the data, a knot is placed at the beginning and end

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of the time break. The interval between these two knots has zero data points
and the interval immediately preceding the time break may have more than
NPTO points but fewer than 2 · NPTO points. If immediately after a time break
there is a second time break before NPTO points have been read, the few (less
than NPTO) points read between the two time breaks are discarded. If a time
break occurs while reading points for the first knot interval, the few (less
than NPTO) points are discarded and the first four knots repositioned at the
first data time after the time break. If a time break occurs during the last
interval, the portion of the last interval contiguous to the previous interval
is kept and the remainder of the points in the last interval are discarded.
If there are at least NPTO points kept, these points form the last interval.
If there are less than NPTO points kept, these points are appended to the pre-
vious interval so that the number of knot intervals is reduced by one. The
time difference between successive data points which is used to define a time
break is named FITBRK. FITBRK is dependent on the sample rate. The time dif-
ference between successive data points used to define a time break in the first
and last knot intervals is FITBRK/5. This smaller value is used in the first
and last interval because it is critical to obtain a good fit in these intervals.
The flow chart on the following pages more clearly defines the logic for select-
ing the knot times. The following define the variables in the flow chart:

NOTS	= number of interior knots	$R(\cdot)$	= array of range measurements
KR	= number of knot intervals	$NPTS(\cdot)$	= array of point counts for knot intervals
$TT(\cdot)$	= array of knot times		
NCOUN	= total data point count	IBRK	= logical denoting the occurrence of a time break
$T(\cdot)$	= array of data times	STA	= data start time
		ETA	= data end time

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IBRK = F
NOTS = 1
NCOUN = 0
NPT   = NPTO
NP   = 0

DO FOR I = 1, N

Read T(I),R(I)>0

IF

NCOUN=0

THEN
TL   = T(I)
TT(K) = T(I),K=1,4
STA   = T(I)

NCOUN=NCOUN+1

IF

T(I)-TL>FITBRK.
AND
NOTS>1

THEN

IF

IBRK=F

THEN

NPTS(NOTS-1)=NPTS(NOTS-1)+NP
TT(NOTS+3) = TL+EPS
IBRK = T
NP   = 1
TSAVE= T(I)
TL   = T(I)

ELSE

NCOUN = NCOUN - NP
TSAVE = T(I)
TL   = T(I)
NP   = 1

ELSE IF

NOTS = 1
AND
T(I)-TL≥.2*FITBRK

THEN

TT(K) = T(I),K=1,4
NP   = 1
NCOUN = 1
STA   = T(I)
TL   = T(I)

ELSE

NP = NP+1

IF

NP = NPT
AND
IBRK=F

THEN

NOTS   = NOTS+1
TT(NOTS+3)   = T(I) + EPS
NPTS(NOTS-1)= NPT
NP   = 0
TL   = T(I)

ELSE IF

NP = NPT
AND
IBRK=T

THEN

NOTS   = NOTS+1
TT(NOTS+3)   = TSAVE
NPTS(NOTS-1) = 0
NOTS   = NOTS+1
TT(NOTS+3)   = T(I)+EPS
NPTS(NOTS-1) = NPT
NP   = 0
IBRK   = F
TL   = T(I)

A

5

A

NOTS ≤ 1

IF

IBRK = T — THEN → ETA = TT(NOTS + 3)
NCOUN = NCOUN - NP

ELSE

TT(NOTS + 3)   = TL + EPS
NPTS (NOTS - 1) = NPTS(NOTS - 1)+ NP
ETA   = TL

KR   = NOTS - 1
INDX = NCOUN - NPTS(KR)
TL   = T(INDX + 1)

DO FOR K = 2, NPTS(KR)

IF   IF   IF

T(INDX + K) ≥ .2*FITBRK — THEN → K-1 ≤ NPTO — THEN → NOTS = NOTS-1 → NOTS ≤ 1 — THEN

ELSE   ELSE   ELSE

NPTS(KR) = K-1   KR = KR - 1
                           NPTS(KR) = NOTS(KR)+K-1

TT(NOTS + 3) = TL + EPS
ETA = TL

TL = T(INDX) + K

6
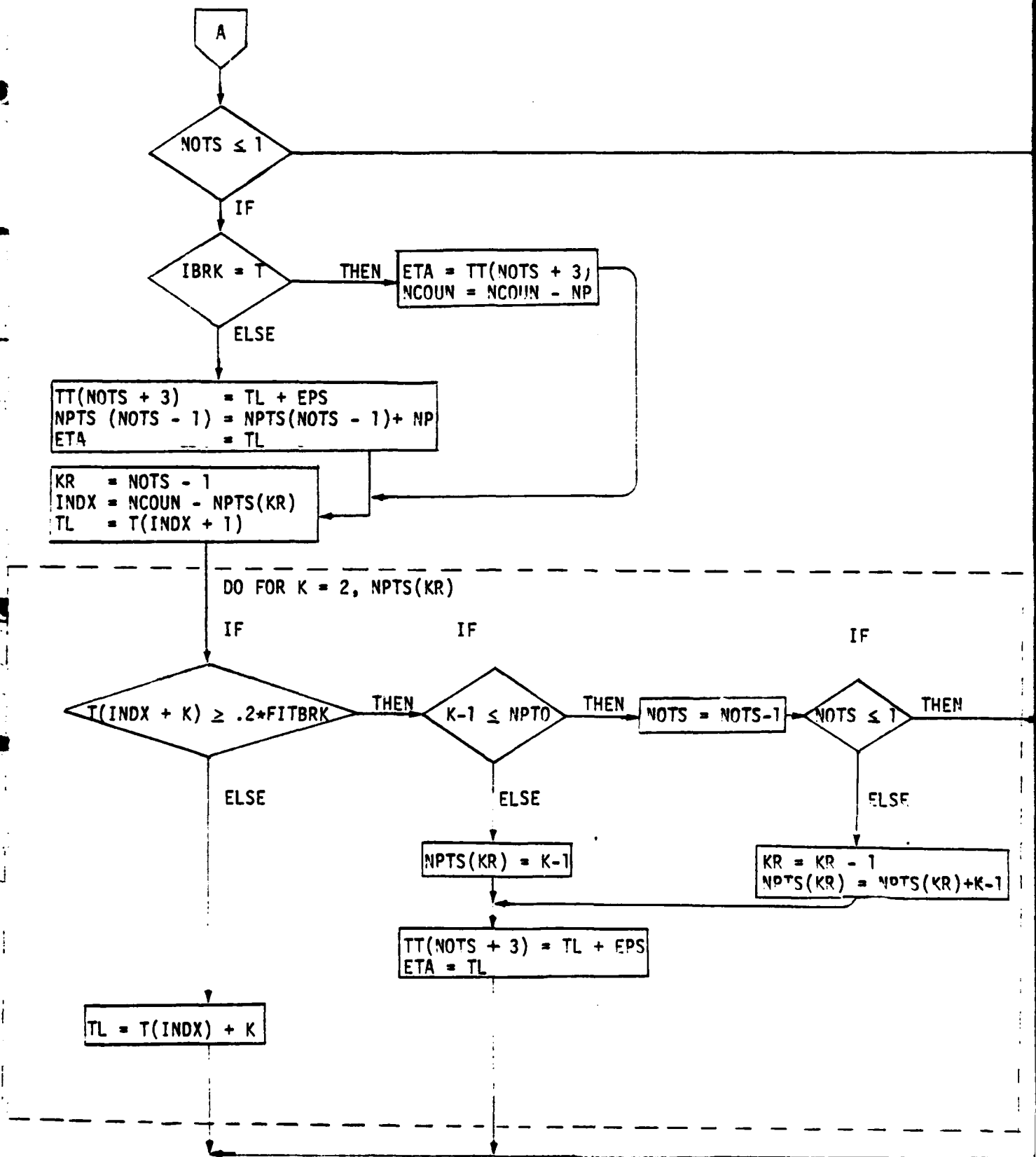
THE LEAST SQUARES NORMAL EQUATIONS

At the $k\underline{\text{th}}$ iteration of the fitting procedure the weighted sum of squares

$$\sum_{j=1}^{N} W_j^{(k)} (R(t_j) - \sum_i b_i^{(k)} B_i(t_j))^2 \tag{5}$$

is minimized. The least squares normal equations are obtained by differentiating (5) with respect to $b_i^{(k)}$. The least squares normal equations are

$$\sum_{j=1}^{N} W_j^{(k)} B(t_j) B^T(t_j) \hat{b}^{(k)} = \sum_{j=1}^{N} W_j^{(k)} B(t_j) R(t_j) \tag{6}$$

where $B^T(t_j)$ is the vector of cubic B-splines

$$B^T(t_j) = [B_1(t_j) \; B_2(t_j) \; - \; - \; - \; B_n(t_j)] \tag{7}$$

Due to the nature of the B-splines the positive definite matrix on the left of (6) is banded with three bands above and below the main diagonal. To conserve storage the four distinct diagonals of this matrix are stored as columns of a vertical matrix. The dimension of the vector $\hat{b}^{(k)}$ is NOTS + 2 where NOTS is the number of interior knots. The banded least squared normal equations are solved by a banded Cholesky decomposition algorithm. The sums of both sides of (6) are performed sequentially so that all of the ranges and weights are not needed in core simultaneously. The IRWLS can be continued for a fixed number of iterations or until the fit has converged.

INITIAL WEIGHTS

In many situations the IRWLS procedure works successfully when all of the initial weights are set to one, i.e., the iteration is started with an ordinary

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unweighted least squares solution. We have found that the use of the unweight-
ed least squares start will usually result in convergence of the IRWLS cubic
spline to a good fit with outliers correctly identified, but that many fewer
iterations are required if a robust choice of initial weights is used. When
outliers are present in either the first or last intervals, the choice of
initial weights in these intervals is most important.

The initial weights for the robust cubic spline fit are chosen on a localized
basis. Let $R(t_i)$, $i = 1$, NPTS(K) be the range measurements in the $K\underline{th}$ knot
interval. To determine the weights $W_i^{(o)}$, $i = 1$, NPTS(K) in the $K\underline{th}$ interval a
linear curve is robustly fitted to the measurements in the interval. Several
methods for robustly fitting the linear curve have been tried, including the
nested median method of Siegel [1], the method of Theil [2], a modified Theil
method, and an M-estimate using a Hampel ψ-function. Most methods performed
about equally well on the data sequences tested. The results of some of these
tests are given in Appendix A. Because of its simplicity, the modified method
of Theil was selected for routine application. This method is described in the
following paragraph.

Let $\bar{R}$ be the median of the observations in the $K\underline{th}$ knot interval.

$$\bar{R} = \text{median } \{R(t_i)\} \atop i = 1, \text{ NPTS}^1(K)$$ (8)

Let $\bar{t}$ be the time corresponding to $\bar{R}$. The median $\bar{R}$ can be represented as the
average of two observations,

$$\bar{R} = (R(t_{m_1}) + R(t_{m_2}))/2$$ (9)

where $m_1 = m_2$ if NPTS(K) is odd. Define the set of slopes $\{S_i\}$

$$S_i = \frac{R(t_i) - \bar{R}}{t_i - \bar{t}} \qquad \begin{array}{l} i = 1, \text{ NPTS}(K) \\ i \neq m_1, m_2 \end{array} \tag{10}$$

Let $\bar{s}$ be the median of the slopes,

$$\bar{s} = \text{median } \{S_i\}$$
$$\begin{array}{l} i = 1, \text{ NPTS}(K) \\ i \neq m_1, m_2 \end{array} \tag{11}$$

Let $\bar{r}_i$ be the residual,

$$\bar{r}_i = R(t_i) - \bar{s}(t_i - \bar{t}), \ i = 1, \text{ NPTS}(K) \tag{12}$$

Let $\bar{r}$ be the median of these residuals,

$$\bar{r} = \text{median } \{\bar{r}_i\}$$
$$i = 1, \text{ NPTS}(K) \tag{13}$$

Now compute the residuals,

$$r_i = \bar{r}_i - \bar{r}, \ i = 1, \text{ NPTS}(K) \tag{14}$$

The initial weights, $W_i^{(o)}$, are computed from these residuals using a Hampel ψ-function.

$$W_i^{(o)} = \frac{\psi\left(\frac{r_i}{s_k}\right)}{\left(\frac{r_i}{s_k}\right)}, \ i = 1, \text{ NPTS}(K) \tag{15}$$

Where s_k is the robust dispersion parameter,

$$s_k = \text{median}_i \ \{|r_i|\}/.6745 \tag{16}$$

Since the main concern in setting the initial weights is to protect the cubic spline fit from the gross outliers, the break points of the Hampel ψ-function are set at $a = 2$, $b = 3$, $c = 4$.

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SOME EXAMPLES

Several hundred data passes have been run with the fitting procedure described. Since there is an average of maybe five receivers on each data pass, the robust preprocessing method described has been used on more than one thousand measurement sequences. The method has performed successfully on all of these sequences. Most of these sequences are rather uneventful, having only a few isolated outliers. There have been some sequences which have some rather dense bursts of outliers. These sequences best illustrate the ability of the method described to detect outliers. Fig 1 presents a range measurement sequence and Fig 2 the robust cubic spline fit to this sequence. Note that the outliers in Fig 1, which have been darkened, occur in many sizes. The outliers at the top of the graph were added by hand since they all occurred far off scale at the top. The sequence of Fig 1 has about 10% outliers. All outliers have been successfully detected and removed by the robust spline fit. The measurements in Fig 1 have two dense burst of outliers, one in the interval (62356.6, 62359.5) and another in the interval (62367.2, 62375.4). The measurement sequence in Fig 3 has outlier bursts in the intervals (62358.4, 62362.6), (62374.4, 62376.4), and (62379.7, 62382.4). The sequence in Fig 3 has about 15% outliers. The sequence in Fig 5 has bursts of outliers during the intervals (63117.8, 63122.6) and (63128.2, 63131.5). Any points away from the main curve should be considered outliers in Figs 1, 3, and 5. Note also in Figs 1, 3, and 5 that there are time breaks in the measurement sequences, another important consideration in preprocessing. The cubic spline fit to the sequence of Fig 1 is given in Fig 2. The cubic spline fit to the sequence of Fig 3 is given in Fig 4 and the cubic spline fit

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to the measurement sequence in Fig 5 is given in Fig 6. The knot intervals
in Figs 2, 4, 6 are designed to contain twenty data points. Note that some
of the time breaks have been filled with fitted data points. The filling
of the time breaks is controlled by the length of the time breaks in
relation to the sample rate and the proportion of outliers found in a
knot interval. The robust cubic spline preprocessor has deleted all
outliers from the measurement sequence, generated measurements during the
time breaks as desired, and synchronized different measurement sequences
if desired. In addition the measurement variances are available for
further processing. The IRWLS cubic spline fit converged in 3 - 4
iterations for the examples displayed. This fairly rapid convergence is
dependent on a robust method for choosing good initial weights. Surprisingly,
the IRWLS cubic spline iteration for these examples also converges using an
unweighted least squares start, but at the expense of more iterations.
For the measurement sequences displayed here the IRWLS cubic spline fit
converged in 7 - 8 iterations using an unweighted least squares start.
Thus, at least in these examples, a good choice of the initial weights
results only in a significant improvement in computing efficiency and not in
an improvement of fit. Besides a good selection of initial weights, another
important choice is the number of data points per knot interval, NPTO.
NPTO must be large enough so that is likely that only a fraction, say less
than one fourth of the data points in any interval will be outliers. On
the other hand, if NPTO is too large, the robust linear curve fit may not
be a good enough representation of the variation of the data in the interval.

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Range - Ft

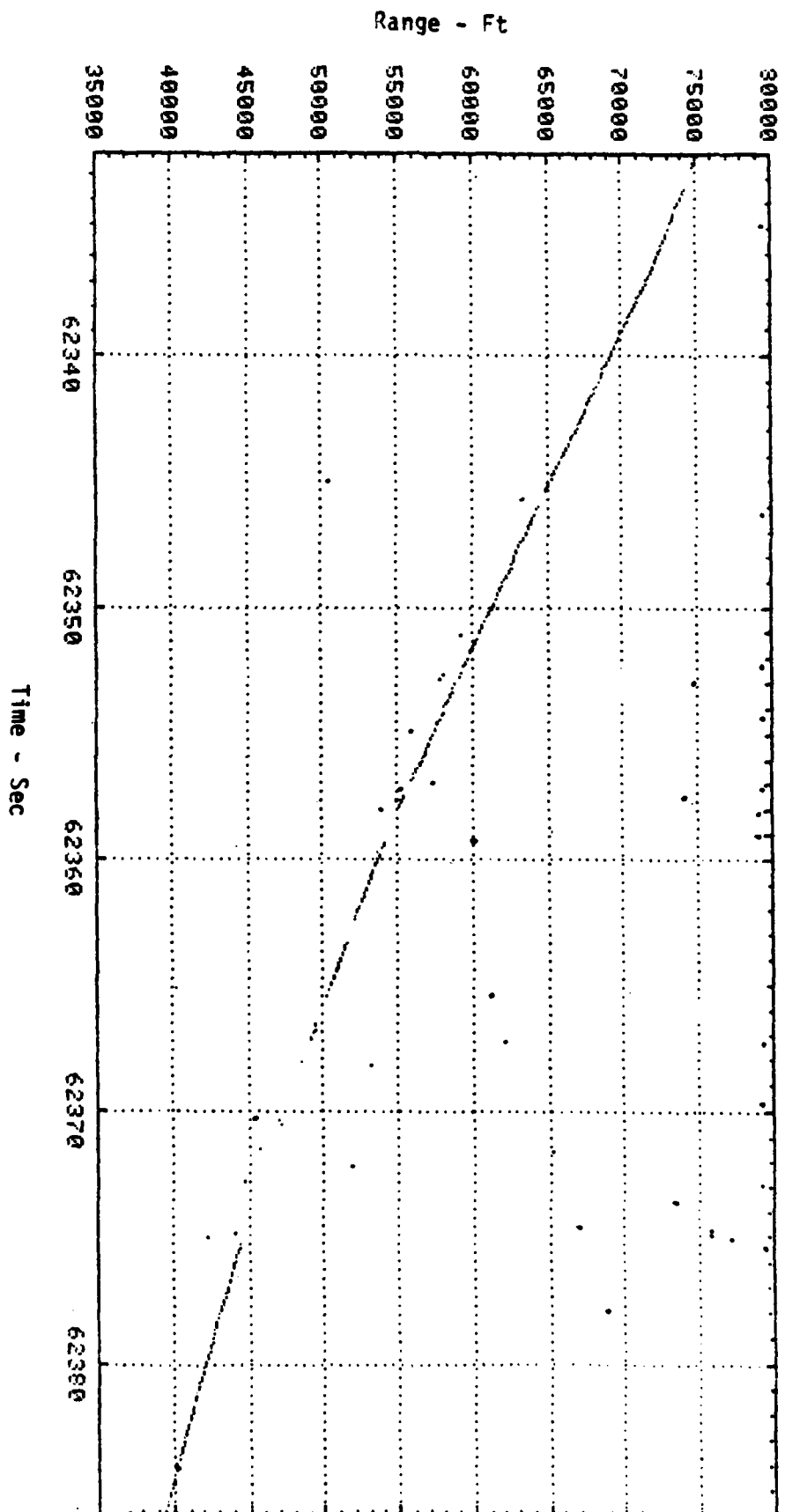Time - Sec

FIGURE 1

12

Range - Ft

Time - Sec

FIGURE 2

13

Range - Ft

Time - Sec

FIGURE 3

14

Range - Ft

40000   45000   50000   55000   60000

62340

62350

62360

Time - Sec

62370

62380

FIGURE 4
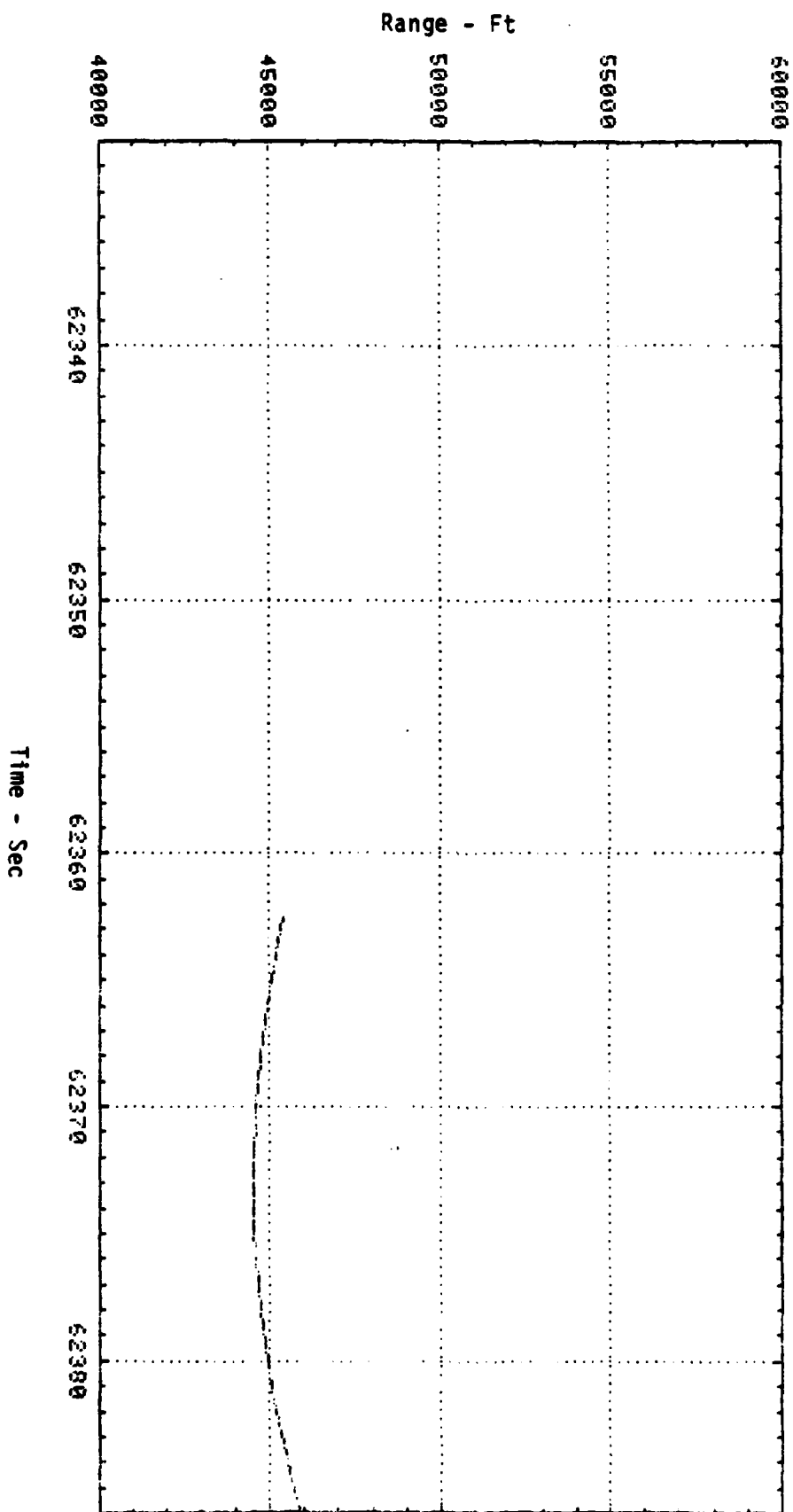
15

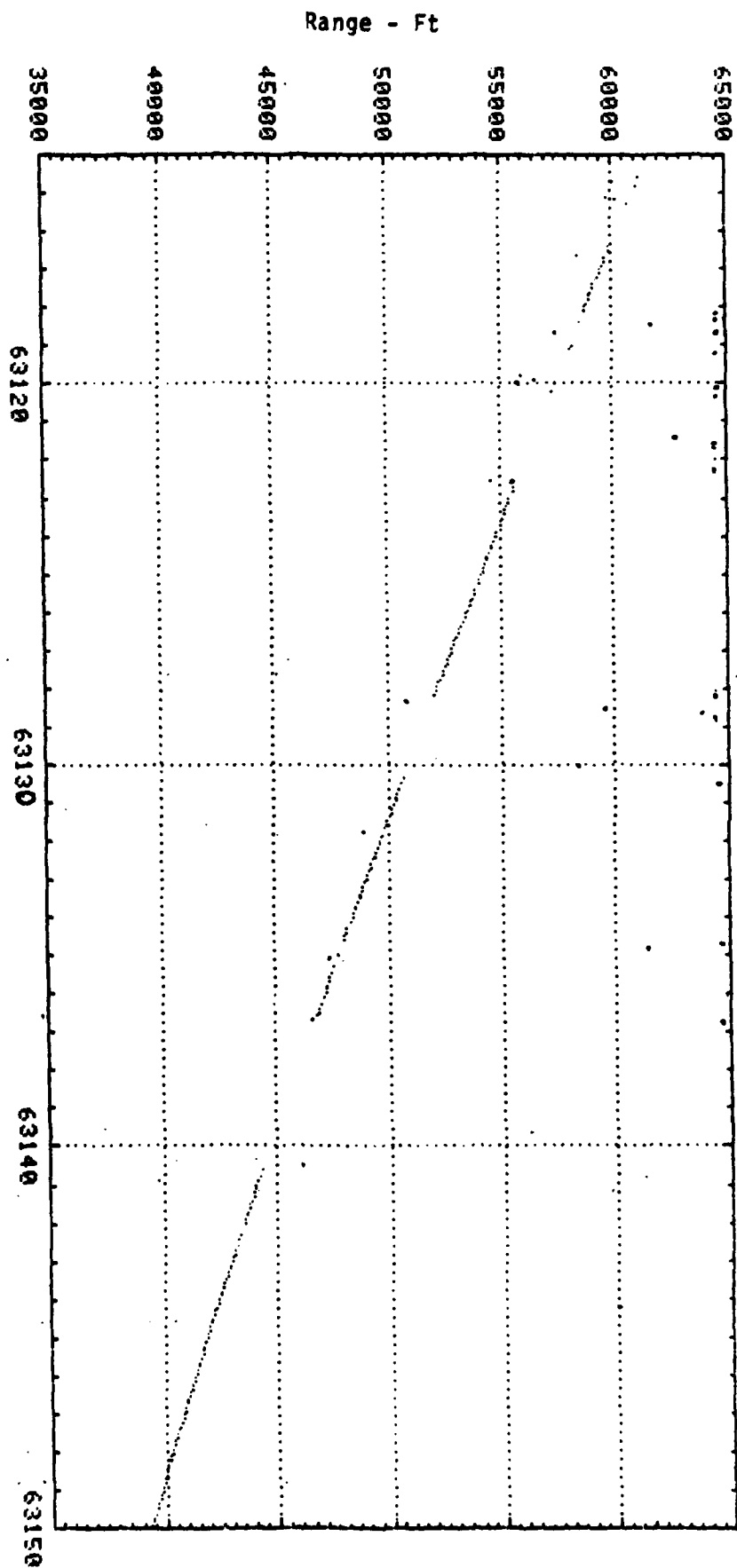Range - Ft

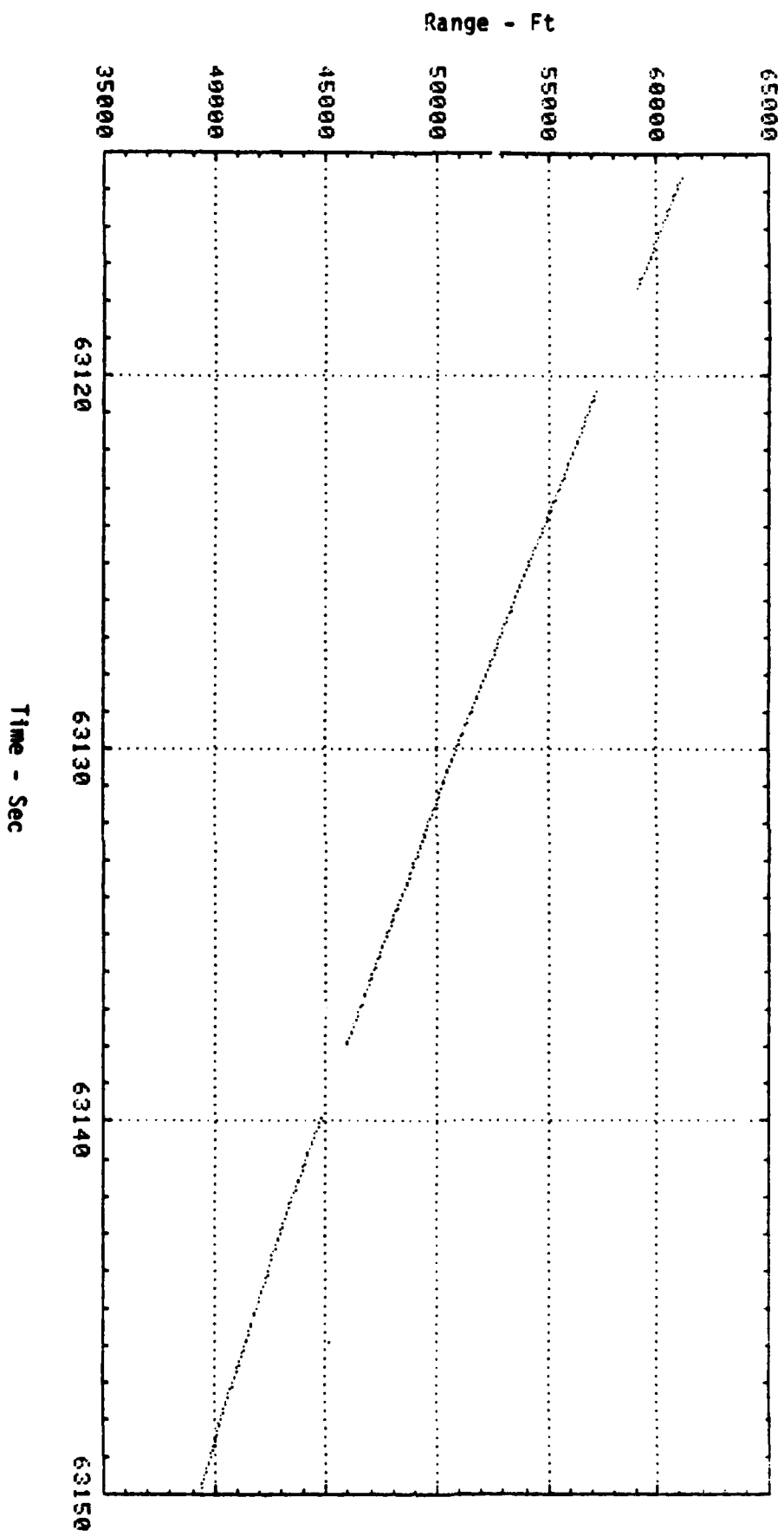FIGURE 5

Time - Sec

16

Range - Ft

Time - Sec

FIGURE 6

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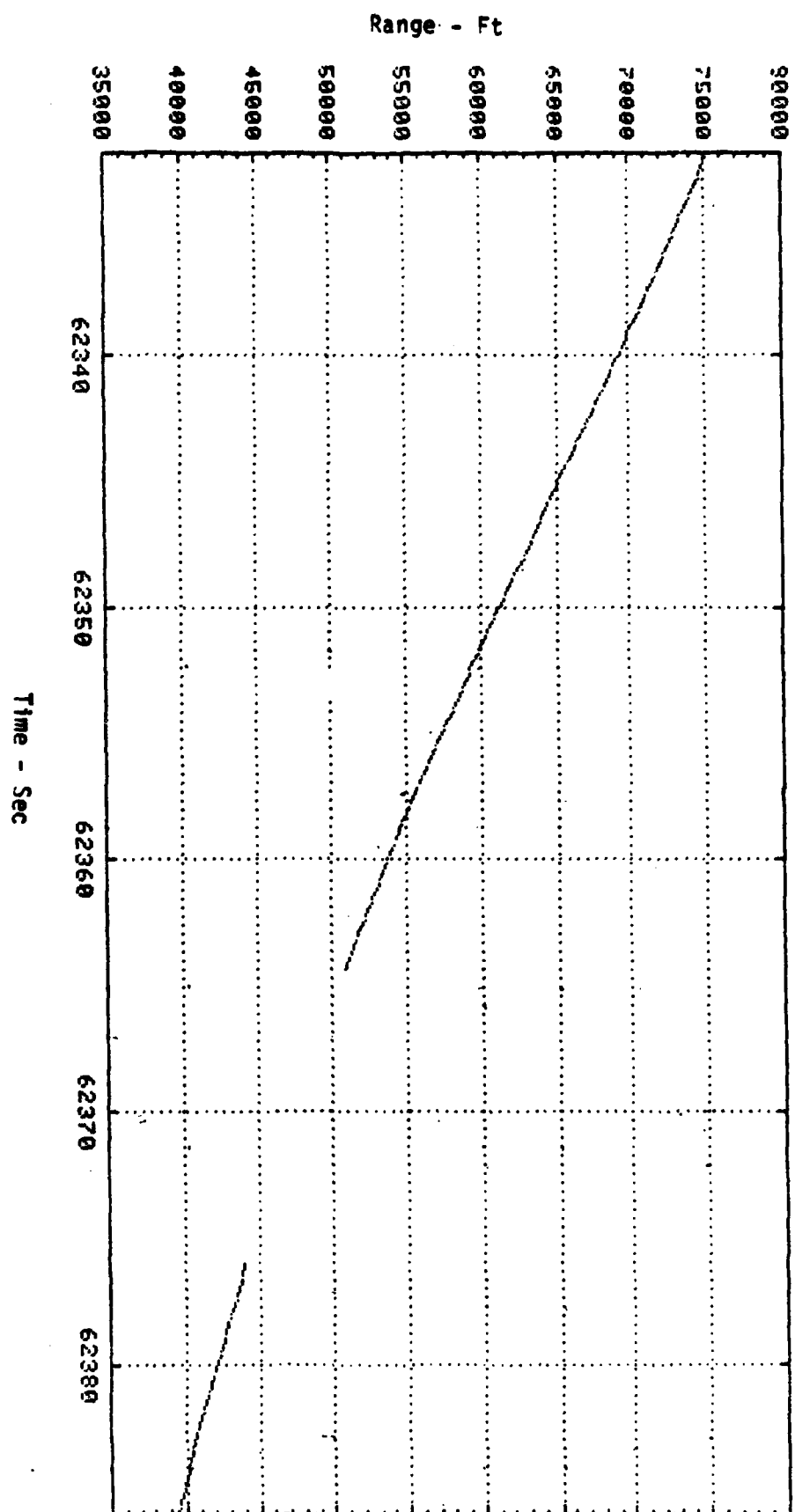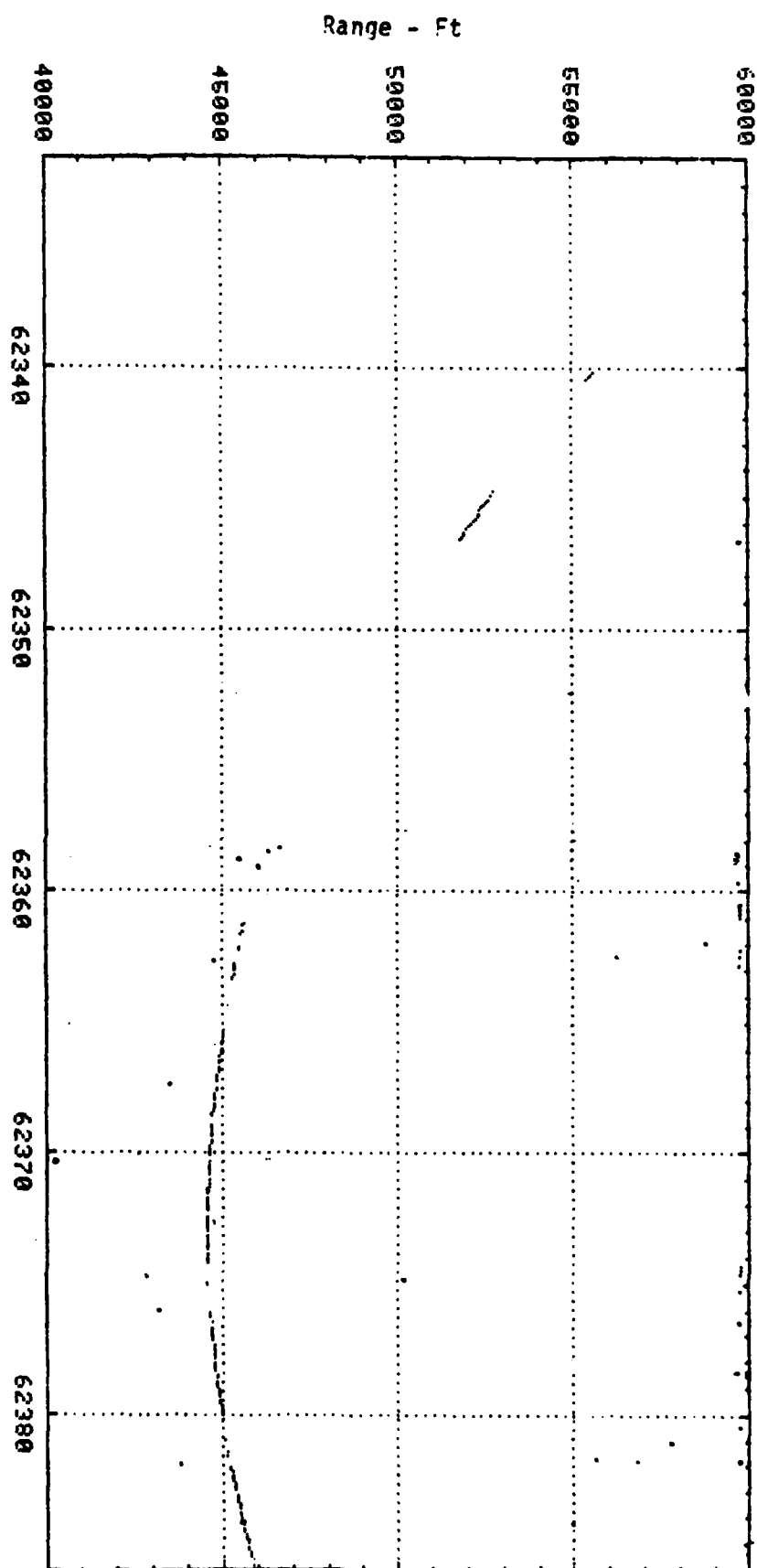
APPENDIX

This appendix describes several methods of choosing the initial weights
for the robust cubic spline preprocessing and compares the results of using
these methods on several data sets. Each of these methods robustly fits a
linear curve in each of the knot intervals and then computes the initial
weights from the curve fit residuals using a Hampel ψ-function. Let
$R(t_i)$, $t_i = 1$, NPTS(k) be the range measurements in the kth knot interval.

Theil Method

Define the slopes s_{ij}

$$S_{ij} = \frac{R(t_j) - R(t_i)}{t_j - t_i} \qquad j>i$$

Let $\bar{s}$ be the median of these slopes,

$$\bar{s} = \underset{\substack{i,j \\ j>i}}{\text{median}} \ \{s_{ij}\}$$

Define the residuals $\bar{r}_i$,

$$\bar{r}_i = R(t_i) - \bar{s} \, t_i$$

Let $\bar{r}$ be the median of the residuals, $\bar{r}_i$

$$\bar{r} = \underset{i=1,\text{NPTS}(k)}{\text{median}} \ \{\bar{r}_i\}$$

Then the residuals $r_i = \bar{r}_i - \bar{r}$ are used to compute the initial weights with a
Hampel ψ-function.

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Nested Medians

Nested or repeated medians is a robust regression method recently described by Siegel [1]. Siegel shows that this method has the highest breakdown method of any known method. This method is particularly easy to apply for a linear fit. It is similar to the Theil method and modified Theil method already described.

Define the slopes s_{ij},

$$s_{ij} = \frac{R(t_j) - R(t_i)}{t_j - t_i} \quad i \neq j \tag{A-1}$$

Define $\bar{s}_i$ by

$$\bar{s}_i = \underset{\substack{j=1,\text{NPTS}(k) \\ j \neq i}}{\text{median}} \{s_{ij}\} \tag{A-2}$$

and further let $\bar{\bar{s}}$ be defined by

$$\bar{\bar{s}} = \underset{i=1,\text{NPTS}(k)}{\text{median}} \{\bar{s}_i\} \tag{A-3}$$

Similarly, let a_{ij} be the intercepts

$$a_{ij} = \frac{R(t_i)t_j - R(t_j)t_i}{t_j - t_i} \quad j \neq i \tag{A-4}$$

Define $\bar{a}_i$ as

$$\bar{a}_i = \underset{\substack{j=1,\ \text{NPTS}(k) \\ j \neq i}}{\text{median}} \{a_{ij}\} \tag{A-5}$$

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and further define $\bar{a}$ by

$$\bar{a} = \underset{i=1,NPTS(k)}{\text{median}} \{\bar{a}_i\} \tag{A-6}$$

Let r_i be the residuals

$$r_i = R(t_i) - \bar{a} - \bar{s} \, t_i, \quad i=1, NPTS(k) \tag{A-7}$$

The weights $W_i^{(0)}$ are computed from these residuals using a Hampel ψ-function.

The following data sets were taken from the knot intervals of the data sequences used previously to illustrate the application of the robust range measurement preprocessing. The first data set, shown in Fig A1 is taken from the measurement sequence given in Fig 1. The measurements in this set are from the time interval 62356.6 - 62359.5. The second data set, shown in Fig A2 is taken from the measurement sequence in Fig 5. The measurements in this set are from the time interval 63128.2 - 63131.6. In each of the data sets the weights are calculated from the residuals r_i by

$$W_i^{(0)} = \frac{\psi(r_{i/s})}{(r_{i/s})} \tag{A-8}$$

where $\psi(\cdot)$ is a Hampel ψ-function with breakpoints 2., 3., 4. In both of those data sets there are eight outliers in the sample of twenty. Each of the robust linear methods seem to have no difficulty in identifying the outliers in these data sets.

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TIME	RANGE	MODIFIED THEIL RES	NESTED MEDIAN RES	THEIL RES	MODIFIED THEIL WEIGHT	NESTED MEDIAN WEIGHT	THEIL WEIGHT
.6	56083	-8	0	5	1	1	1
.7	56004	-11	-5	0	1	1	1
.8	55938	-2	2	7	1	1	1
.9	57336	1471	1473	1478	1	1	1
1.0	97546	41757	41757	41761	0	0	0
1.1	74022	18308	18307	18311	0	0	0
1.2	55190	-447	-451	-447	0	0	0
1.4	55478	-8	-15	-12	1	1	1
1.5	55400	-11	-20	-16	1	1	1
1.6	55321	-14	-25	-22	1	1	1
1.7	55249	-11	-23	-21	1	1	1
1.8	55170	-14	-29	-26	1	1	1
1.9	55111	2	-14	-12	1	1	1
2.0	54009	-1024	-1042	-1040	0	0	0
2.1	99357	44399	44379	44380	0	0	0
3.1	172631	118428	118389	118389	0	0	0
3.2	59980	5853	5812	5811	0	0	0
3.3	54094	42	0	-1	1	1	1
3.4	54022	46	1	0	1	1	1
3.5	53950	49	3	1	1	1	1

FIGURE A1

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TIME	RANGE	MODIFIED THEIL RES	NESTED MEDIAN RES	THEIL RES	MODIFIED THEIL WEIGHT	NESTED MEDIAN WEIGHT	THEIL WEIGHT
.2	94462	43071	43013	42937	0	0	0
.3	50859	-465	-521	-594	0	0	0
.5	59606	8411	8362	8297	0	0	0
.6	63825	12696	12649	12588	0	0	0
.7	122290	71226	71182	71125	0	0	0
1.0	58359	7491	7456	7410	0	0	0
1.1	79974	29172	29139	29097	0	0	0
1.3	50699	-2	-29	-64	1	1	1
1.4	50584	-21	-45	-77	1	1	1
1.5	169737	119196	119175	119148	0	0	0
1.6	50472	-3	-21	-45	1	1	1
1.7	50387	-22	-38	-58	1	1	1
1.8	50335	-9	-22	-38	1	1	1
1.9	50275	-3	-13	-26	1	1	1
2.0	50223	9	2	-6	1	1	1
2.1	50131	-17	-21	-26	1	1	1
2.2	50085	2	-3	0	1	1	1
2.3	50013	-4	-3	0	1	1	1
2.4	49947	-5	-1	6	1	1	1
2.5	49901	14	26	32	1	1	1

FIGURE A2

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